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Applications of the Invariance Principle for Compact Processes II. Asymptotic Behavior of Solutions of a Hyperbolic Conservation Law*

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1. INTRODUCTION

The intent of this work is to provide a demonstration of the applicability of the theory of uniform processes, developed in [1], in the study of asymptotic stability of solutions of evolutionary equations. Specifically, we consider the asymptotic behavior, as $t \to \infty$, of solutions of the initial value problem for the conservation law

$$v_t + f(v)_x = 0 \tag{1.1}$$

under periodic initial conditions.

Lax [2] shows that if f is strictly convex or concave, solutions tend to a constant state as $t \to \infty$. Here we establish a similar result under the sole assumption that the set of points of inflection of f has no accumulation point on the real line.[1] To this end, we prove that (1.1) generates a uniform dynamical system. A uniform dynamical system is the simplest example of a uniform process [1]. The characteristic property is that motions are positively Liapunov stable with respect to the phase space. In the present example the natural phase space is a closed bounded subset of L^∞ equipped with the metric induced by the norm of L^1_{loc}.

From the point of view of applications to asymptotic stability theory, the principal feature of uniform processes is that the standard continuity requirement on Liapunov functionals can be substituted by a mere semicontinuity condition. This turns out to be a great advantage here because the natural

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[1] Some assumption of this nature is necessary since it is clear that solutions of (1.1) with f linear do not tend in general to a constant state as $t \to \infty$.

Liapunov functional $\operatorname{ess\,sup}_{-\infty<x<\infty} v(x, t)$ is not continuous on the phase space but merely lower semicontinuous.

We show that the orbit of any motion of the dynamical system is precompact in the phase space. Then the asymptotic behavior of motions is established by employing the information that the ω-limit set is invariant and the Liapunov functional is constant on it.

2. The Invariance Principle for Uniform Dynamical Systems

The properties of uniform dynamical systems recorded below can be derived by a specialization of the theory of uniform processes developed in [1].

Let X be a complete metric space with metric d, R the set of real numbers, and R^+ the set of nonnegative numbers.

Definition 2.1. A *uniform dynamical system* on X is a map

$$u : X \times R^+ \to X$$

with the following properties:

(i) $\quad u(\chi, 0) = \chi \qquad$ for all $\chi \in X$. $\qquad$ (2.1)

(ii) $\quad u(\chi, \sigma + \tau) = u(u(\chi, \sigma), \tau) \qquad$ for all $\chi \in X$, $\sigma, \tau \in R^+$. $\qquad$ (2.2)

(iii) The one-parameter family of maps

$$u(\cdot, \tau) : X \to X, \qquad \text{parameter } \tau \in R^+,$$

is equicontinuous.

Definition 2.2. Let u be a uniform dynamical system on X and $\chi \in X$. Then

(i) The *motion* through χ is the map $u(\chi, \cdot) : R^+ \to X$.

(ii) The *orbit* of the motion through χ is the range of the above map.

(iii) The *ω-limit set* of the motion through χ is the set

$$\omega(\chi) = \bigcap_{\sigma \geqslant 0} Cl_X \bigcup_{\tau \geqslant \sigma} u(\chi, \tau). \tag{2.3}$$

Remark 2.1. Equivalently, a uniform dynamical system is a dynamical system in the sense of [3, Def. 4.1] whose motions are positively Liapunov stable with respect to X [4, Chap. V, Def. 8.03]. The assumption that motions are continuous which is a part of the standard definition of a dynamical system (e.g. [5, Def. 1]) is not necessary in the present discussion.

Remark 2.2. If for some $\chi \in X$ the orbit of the motion through χ is precompact in X, then $\omega(\chi)$ is nonempty, compact and

$$u(\chi, \tau) \to \omega(\chi), \qquad \tau \to \infty. \tag{2.4}$$

DEFINITION 2.3. Let u be a uniform dynamical system on X. A map $V : X \to R$ is called a *semicontinuous Liapunov functional* for u if

(i) $$V(u(\chi, \tau)) \leqslant V(\chi) \qquad \text{for all } \chi \in X,\ \tau \in R^+. \tag{2.5}$$

(ii) For any convergent sequence $\{\chi_n\}$ in X,

$$V(\lim_{n\to\infty} \chi_n) \leqslant \liminf_{n\to\infty} V(\chi_n). \tag{2.6}$$

PROPOSITION 2.1. *Let u be a uniform dynamical system on X and V a semicontinuous Liapunov functional for u. Suppose that for some $\chi \in X$, $\omega(\chi)$ is nonempty and $\psi \in \omega(\chi)$. Then*

$$V(u(\psi, \tau)) = V(\psi) \qquad \textit{for all } \tau \in R^+. \tag{2.7}$$

3. Asymptotic Behavior of Solutions of a Conservation Law under Periodic Initial Conditions

Consider the initial value problem

$$v_t + f(v)_x = 0, \qquad (x, t) \in R \times R^+, \tag{3.1}$$

$$v(x, 0) = v_0(x), \qquad x \in R, \tag{3.2}$$

where f is continuously differentiable and $v_0(\cdot)$ is measurable and essentially bounded on R,

$$m \leqslant v_0(x) \leqslant M, \qquad \text{a.e. on } R. \tag{3.3}$$

In general, there is no classical solution and there is more than one weak solution of (3.1), (3.2). Alternative albeit equivalent characterizations of admissible weak solutions have been proposed by Kružkov [6] and Hopf [7]. Following Hopf,

DEFINITION 3.1. A bounded and measurable function $v(x, t)$ on $R \times R^+$ will be called an *admissible weak solution* of (3.1), (3.2) if for any convex function $g(v)$ and any smooth nonnegative function $\phi(x, t)$ with compact support on $R \times R^+$,

$$\int_0^\infty \int_{-\infty}^\infty [g(v)\,\phi_t + F(v)\,\phi_x]\, dx\, dt + \int_{-\infty}^\infty g(v_0)\,\phi(x, 0)\, dx \geqslant 0, \tag{3.4}$$

where

$$F(v) \equiv \int_0^v f'(\xi)\, dg(\xi). \tag{3.5}$$

Kružkov [6] proves the following existence theorem:

PROPOSITION 3.1. *There exists a unique admissible weak solution* $v(x, t)$ *of* (3.1), (3.2). *Furthermore,* $v(\cdot, t) \in C^0[R^+; L^1_{\mathrm{loc}}(R)]$ *and*

$$v(x, 0) = v_0(x), \text{ a.e. } on\ R. \tag{3.6}$$

In case $v_0(x)$ is of locally bounded variation, the following more precise result has been established (e.g. [8] and the references given there):

PROPOSITION 3.2. *Let* $v(x, t)$ *be the admissible weak solution of* (3.1), (3.2), *where* $v_0(x)$ *is of locally bounded variation on* R, *continuous on the left and satisfies* (3.3). *Then, for each* $t \in R^+$, $v(\cdot, t)$ *is of locally bounded variation, continuous on the left and for any* $-\infty < x_1 < x_2 < \infty$,

$$\operatorname*{Var}_{[x_1,x_2]} v(x, t) \leqslant \operatorname*{Var}_{[x_1-Kt,x_2+Kt]} v_0(x) \tag{3.7}$$

with

$$K \equiv \max_{[m,M]} |f'(\cdot)|. \tag{3.8}$$

The following properties of admissible weak solutions are established in [6]:

PROPOSITION 3.3. *Let* $v(x, t)$, $w(x, t)$ *be the admissible weak solutions of* (3.1) *with initial conditions* $v_0(x)$, $w_0(x)$, *respectively, which satisfy* (3.3). *Then*

(i) *For any* $-\infty < x_1 < x_2 < \infty$ *and any* $t \in R^+$,

$$\int_{x_1}^{x_2} | v(x, t) - w(x, t)|\, dx \leqslant \int_{x_1-Kt}^{x_2+Kt} | v_0(x) - w_0(x)|\, dx. \tag{3.9}$$

(ii) *If* $v_0(x) \leqslant w_0(x)$, $x \in R$, *then*

$$v(x, t) \leqslant w(x, t), \qquad (x, t) \in R \times R^+. \tag{3.10}$$

In particular, it follows that for any $t \in R^+$,

$$\operatorname*{ess\,sup}_{-\infty<x<\infty} v(x, t) \leqslant \operatorname*{ess\,sup}_{-\infty<x<\infty} v_0(x), \tag{3.11}$$

$$\operatorname*{ess\,inf}_{-\infty<x<\infty} v(x, t) \geqslant \operatorname*{ess\,inf}_{-\infty<x<\infty} v_0(x). \tag{3.12}$$

We now restrict our attention to solutions of (3.1), (3.2) where $v_0(\cdot)$ is periodic with period $p > 0$. It is clear that in this case, and for each fixed $t \in R^+$, $v(\cdot, t)$ will also be periodic with period p. Recalling Definition 3.1 it is easy to prove[2] the following result:

PROPOSITION 3.4. *Let $v(x, t)$ be the admissible weak solution of* (3.1), (3.2) *where $v_0(\cdot)$ is periodic with period p. Then, for any convex function g,*

$$\lim_{S\to\infty} \frac{1}{2S} \int_{-S}^{S} g(v(x, t))\, dx \leqslant \lim_{S\to\infty} \frac{1}{2S} \int_{-S}^{S} g(v_0(x))\, dx, \qquad t \in R^+. \tag{3.13}$$

In particular, choosing $g(v) = \pm v$,

$$\lim_{S\to\infty} \frac{1}{2S} \int_{-S}^{S} v(x, t)\, dx = \lim_{S\to\infty} \frac{1}{2S} \int_{-S}^{S} v_0(x)\, dx, \qquad t \in R^+. \tag{3.14}$$

By X we denote the set of measurable functions $v_0(\cdot)$ on R which are periodic with period $p > 0$, satisfy (3.3) and have mean value

$$\lim_{S\to\infty} \frac{1}{2S} \int_{-S}^{S} v_0(x)\, dx = \frac{1}{p} \int_{0}^{p} v_0(x)\, dx = A, \tag{3.15}$$

where A a fixed constant in $[m, M]$. For v_0, $w_0 \in X$ we set

$$d(v_0, w_0) \equiv \lim_{S\to\infty} \frac{1}{2S} \int_{-S}^{S} | v_0(x) - w_0(x)|\, dx = \frac{1}{p} \int_{0}^{p} | v_0(x) - w_0(x)|\, dx. \tag{3.16}$$

It is clear that (X, d) is a complete metric space.

Let $v(x, t)$ be the admissible weak solution of (3.1), (3.2) with $v_0 \in X$. Recalling (3.11), (3.12) and (3.14) we deduce that for any $t \in R^+$, $v(\cdot, t)$ can be identified with a point in X which will be denoted by $u(v_0, t)$.

PROPOSITION 3.5. *The map $u : X \times R^+ \to X$ is a uniform dynamical system.*

Proof. Obviously,

$$u(v_0, 0) = v_0 \qquad \text{for all } v_0 \in X, \tag{3.17}$$

$$u(v_0, \sigma + \tau) = u(u(v_0, \sigma), \tau) \qquad \text{for all } v_0 \in X,\ \sigma, \tau \in R^+. \tag{3.18}$$

Suppose now that v_0, $w_0 \in X$. On account of (3.9) and (3.16),

$$d(u(v_0, \tau), u(w_0, \tau)) \leqslant d(v_0, w_0) \qquad \text{for all } \tau \in R^+, \tag{3.19}$$

which shows that the one-parameter family of maps $u(\cdot, \tau) : X \to X$ is equicontinuous. Thus (Definition 2.1) u is a uniform dynamical system on X.

[2] The proof closely parallels that of Theorem 6.1 in [2].

PROPOSITION 3.6. *The orbit of any motion of u is precompact in X.*

Proof. Fix any $v_0 \in X$ and $\Delta x \in R$. Applying (3.9) for $w_0(x) \equiv v_0(x + \Delta x)$, which in turn implies $w(x, t) = v(x + \Delta x, t)$, and using periodicity,

$$\begin{aligned}\frac{1}{p}\int_0^p |v(x + \Delta x, t) - v(x, t)|\,dx &= \lim_{S\to\infty} \frac{1}{2S}\int_{-S}^{S} |v(x + \Delta x, t) - v(x, t)|\,dx \\ &\leqslant \lim_{S\to\infty} \frac{1}{2S}\int_{-S}^{S} |v_0(x + \Delta x) - v_0(x)|\,dx \\ &= \frac{1}{p}\int_0^p |v_0(x + \Delta x) - v_0(x)|\,dx. \end{aligned} \tag{3.20}$$

It follows that the one-parameter family of functions $v(\cdot, t)$, parameter $t \in R^+$, is $L^1_{\text{loc}}(R)$-equicontinuous and hence precompact in X.

PROPOSITION 3.7. *For $v_0 \in X$ we set*

$$V(v_0) \equiv \operatorname*{ess\,sup}_{-\infty<x<\infty} v_0(x). \tag{3.21}$$

Then $V : X \to R$ is a semicontinuous Liapunov functional for u.

Proof. It is easy to show that if $\{v_n(x)\}$ is a sequence converging to $v_0(x)$ in $L^1_{\text{loc}}(R)$, then $\text{ess sup}_{-\infty<x<\infty} v_0(x) \leqslant \liminf_{n\to\infty}[\text{ess sup}_{-\infty<x<\infty} v_n(x)]$, i.e., (2.6) is satisfied. Furthermore, for any $t \in R^+$ and on account of (3.11), (3.21),

$$V(u(v_0, t)) = \operatorname*{ess\,sup}_{-\infty<x<\infty} v(x, t) \leqslant \operatorname*{ess\,sup}_{-\infty<x<\infty} v_0(x) = V(v_0). \tag{3.22}$$

Therefore, (Definition 2.3) V is a semicontinuous Liapunov functional for u.

We are now prepared to state and prove the main result of this article:

PROPOSITION 3.8. *Let $v(x, t)$ be the admissible weak solution of* (3.1), (3.2) *where $v_0(\cdot)$ is periodic with period p, satisfies* (3.3) *and is of locally bounded variation. Assume further that the function f' attains its relative extrema on a set which has no accumulation point on the real line. Then*

$$v(x, t) \xrightarrow{L^1_{\text{loc}}(R)} A, \qquad t \to \infty, \tag{3.23}$$

where A is the mean value of $v_0(\cdot)$.

Proof. By virtue of Proposition 3.6 and Remark 2.2, we have to show that if w_0 is a point in the ω-limit set of the motion of u through v_0, then $w_0(x) = A$ on R. Since $w_0 \in X$, it suffices to prove that $w_0(x) = \text{constant}$ on R.

Proposition 3.2 together with periodicity implies that for any $t \in R^+$,

$$\frac{1}{p} \operatorname*{Var}_{[0^+, p]} v(x, t) = \lim_{S \to \infty} \frac{1}{2S} \operatorname*{Var}_{[-S, S]} v(x, t) \leqslant \lim_{S \to \infty} \frac{1}{2S} \operatorname*{Var}_{[-S, S]} v_0(x) = \frac{1}{p} \operatorname*{Var}_{[0^+, p]} v_0(x) \tag{3.24}$$

so that, by Helly's theorem, $w_0(x)$ can be identified with a function of locally bounded variation which is continuous on the left.

Using Propositions 3.7 and 2.1 we conclude that if $w(x, t)$ is the admissible weak solution of (3.1) with initial condition $w_0(x)$,

$$\operatorname*{ess\,sup}_{-\infty < x < \infty} w(x, t) = \operatorname*{ess\,sup}_{-\infty < x < \infty} w_0(x) \equiv C \qquad \text{for all } t \in R^+. \tag{3.25}$$

We claim that (3.25) implies $w_0(x) =$ constant on R. Indeed, suppose that this is not the case. Then there is $\epsilon > 0$ and a point $b \in R$ with

$$w_0(b) \leqslant C - 2\epsilon. \tag{3.26}$$

Since $w_0(x)$ is continuous on the left, it follows that

$$w_0(x) \leqslant C - \epsilon \qquad \text{for} \quad a < x \leqslant b. \tag{3.27}$$

On the other hand, there is $\delta > 0$ such that f' is strictly monotone on $(C - \delta, C]$. We now set

$$c \equiv \max\{C - \epsilon, C - \delta\} \tag{3.28}$$

and we define

$$\bar{w}_0(x) \equiv \begin{cases} c & \text{for} \quad x \in (a + np, b + np], \qquad n = 0, \pm 1, \pm 2, \ldots \\ C & \text{otherwise.} \end{cases} \tag{3.29}$$

Let $\bar{w}(x, t)$ be the admissible weak solution of (3.1) with initial condition $\bar{w}_0(x)$. It is clear that $w_0(x) \leqslant \bar{w}_0(x)$ on R so that by Proposition 3.3,

$$w(x, t) \leqslant \bar{w}(x, t) \qquad \text{for all } (x, t) \in R \times R^+. \tag{3.30}$$

Combining (3.30) with (3.25) we deduce

$$\operatorname*{ess\,sup}_{-\infty < x < \infty} \bar{w}(x, t) \geqslant C \qquad \text{for all } t \in R^+. \tag{3.31}$$

Now $\bar{w}(x, t)$ can be constructed explicitly. For definiteness, we assume that f' is strictly decreasing on $(c, C]$. (The discussion in the case where f' is strictly increasing follows similar lines.) For small t $\bar{w}(x, t)$ is formed by a system of simple waves, centered at $(a + np, 0)$, $n = 0, \pm 1, \pm 2, \ldots$, and a system of shocks originating at $(b + np, 0)$, $n = 0, \pm 1, \pm 2,...$, and propagating with speed

$$s \equiv \frac{f(C) - f(c)}{C - c}, \qquad f'(c) > s > f'(C). \tag{3.32}$$

We study the interaction of these waves. We set

$$T_1 \equiv \frac{p - b + a}{s - f'(C)}, \qquad T_2 \equiv \frac{b - a}{f'(c) - s} \tag{3.33}$$

and we distinguish the following two cases:

(i) $T_1 < T_2$. The first interaction will occur at time $T = T_1$ when the shock originating at $(b + np, 0)$ will overcome the simple wave centered at $(a + (n + 1)p, 0)$. The shock will accelerate and will weaken. (Later on the shock will be overcome by the wave centered at $(a + np, 0)$ but we are not interested in this interaction).

(ii) $T_2 \leqslant T_1$. The first interaction will occur at time T_2 when the simple wave centered at $(a + np, 0)$ will overcome the shock originating at $(b + np, 0)$. The schock will weaken and will decelerate. Even so at some time $T \geqslant T_2$ the shock will overcome the simple wave centered at $(a + (n + 1)p, 0)$.[3]

In either case, for $t > T$,

$$\operatorname*{ess\,sup}_{-\infty<x<\infty} \bar{w}(x, t) < C \tag{3.34}$$

which is a contradiction to (3.31). The proof is complete.

Remark 3.1. In the place of (3.21) and on account of (3.12) we could have used equally well the semicontinuous Liapunov functional

$$V(v_0) \equiv -\operatorname*{ess\,inf}_{-\infty<x<\infty} v_0(x). \tag{3.35}$$

Let Y be the set of functions in X which are of locally bounded variation with mean variation uniformly bounded by a constant, say B. By Helly's theorem Y is closed in X while by (3.24) Y is positively invariant relative

[3] Otherwise $\operatorname{ess\,inf}_{-\infty<x<\infty} \bar{w}(x, t) \to C$ as $t \to \infty$ which is impossible since for each fixed $t \in R^+$ the mean value of $\bar{w}(x, t)$ should be equal to

$$[c(b - a) + C(p - b + a)]/p < C.$$

to u. Thus the restriction of u on $Y \times R^+$ is a uniform dynamical system on Y. A natural Liapunov functional in this case is the mean variation

$$V(v_0) = \lim_{S\to\infty} \frac{1}{2S} \operatorname*{Var}_{[-S,S]} v_0(x) = \frac{1}{p} \operatorname*{Var}_{[0^+,p]} v_0(x) \tag{3.36}$$

which is also semicontinuous.

It is even possible to construct continuous Liapunov functionals for u. For example from Proposition 3.4 it follows that for any convex function g,

$$V(v_0) = \lim_{S\to\infty} \frac{1}{2S} \int_{-S}^{S} g(v_0(x))\, dx = \frac{1}{p} \int_0^p g(v_0(x))\, dx \tag{3.37}$$

is a continuous Liapunov functional for u.

The problem with Liapunov functionals (3.36), (3.37) is that it is not easy to show that $V(u(w_0, t)) = V(w_0)$ implies $w_0(x) =$ constant on R.

Remark 3.2. If f is strictly convex or strictly concave, Lax [2] establishes the more precise result

$$\operatorname*{ess\,sup}_{-\infty<x<\infty} | v(x, t) - A | = O\left(\frac{1}{t}\right), \qquad \text{as} \quad t \to \infty; \tag{3.38}$$

he also derives the asymptotic shape of $v(x, t)$ as $t \to \infty$.

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